Introduction to MHD dynamos

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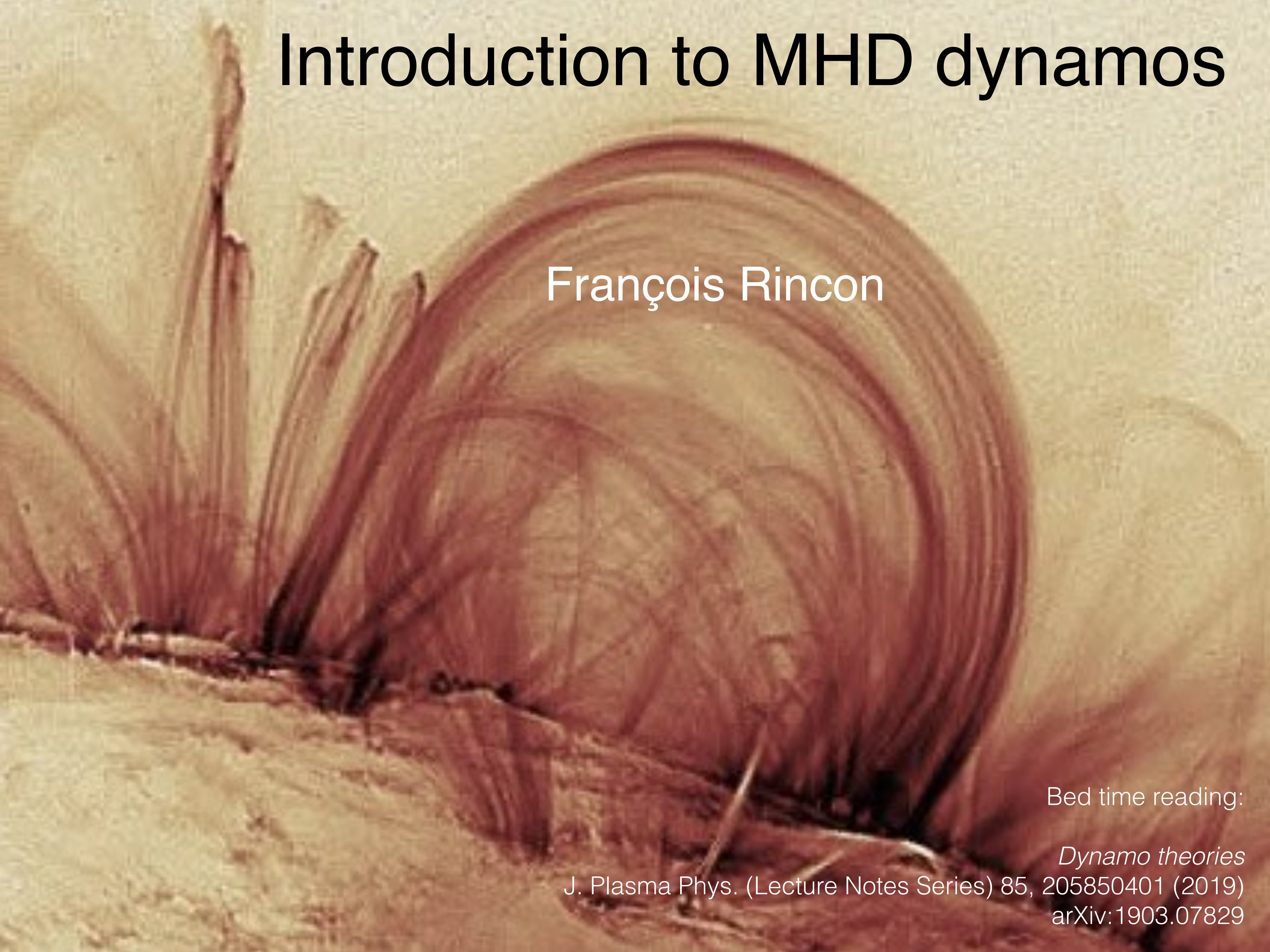


Bed time reading:

Dynamo theories
J. Plasma Phys. (Lecture Notes Series) 85, 205850401 (2019)
arXiv:1903.07829

Tutorial outline

- Context
  - Short and easy (3h)
- Setting the theory stage
  - Not too long and straightforward (4h)
- Small scale MHD dynamos
  - Long and difficult (6h)

Today

- Large-scale MHD dynamos
  - Just a tad shorter, a bit less difficult (4h)
- Connections between large & small-scale dynamos
  - Short and controversial, also difficult (2h)
- Instability-driven MHD dynamos
  - Short and seemingly easier, but actually differently difficult (3h)
- Collisionless plasma dynamo
  - Short and a bit crazy, and even more difficult (4h)

Second session
(February, or whenever you want)

What is dynamo theory about ?

- The origin, and sustainment, of magnetic fields in the universe
  - on the Earth, other planets and their satellites (“planetary magnetism”)
  - in the Sun and other stars (“stellar magnetism”)
  - in galaxies, clusters and the early universe (“cosmic magnetism”)
- Understanding their structural, statistical, and dynamical properties
- Addressing important physics (and maths) problems
  - Deep connections with hydrodynamic turbulence and more generally (turbulent) transport problems
- Coming up with “useful stuff” for experimentalists and observers
  - Warning: people strongly disagree on the definition of “useful stuff”

The fluid/plasma dynamo conundrum

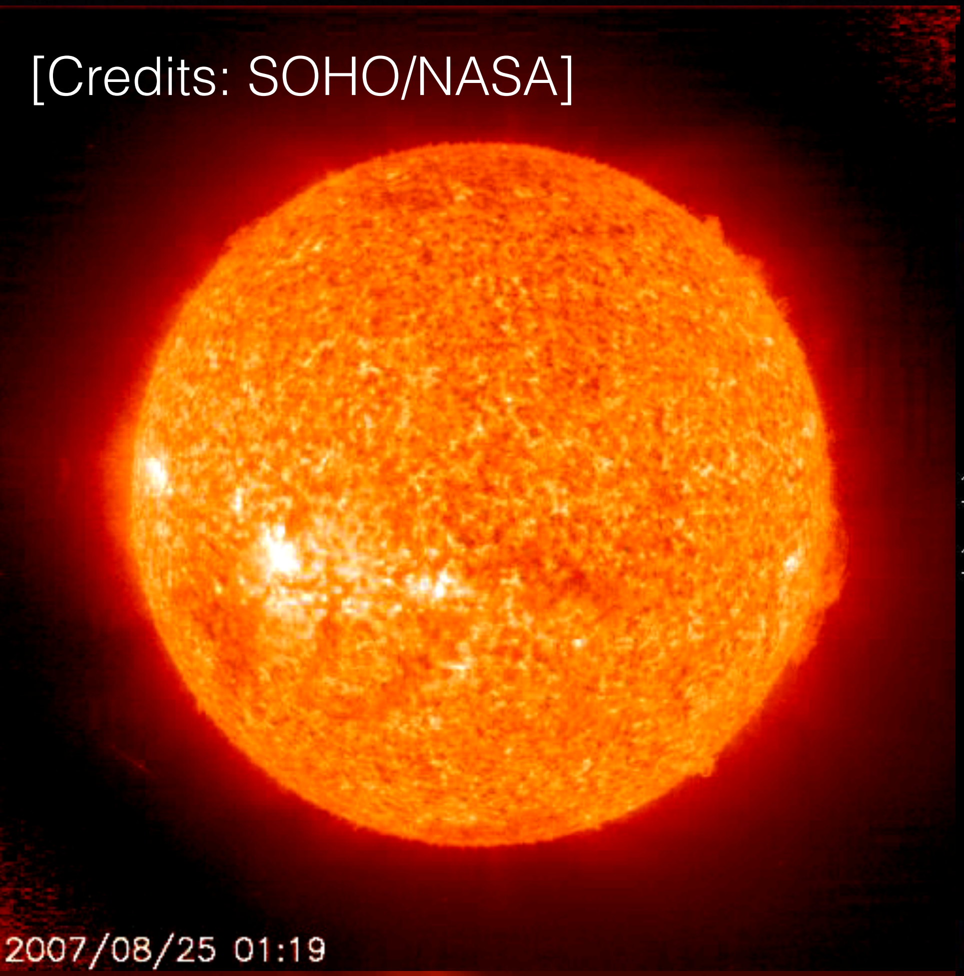
- Most astrophysical bodies, and many planetary interiors, are
  - in an electrically conducting fluid (MHD) or weakly-collisional plasma state
  - in a turbulent state
  - (differentially) rotating: shearing, Coriolis and precessing effects
- Main questions
  - Can flows of electrically conducting fluid/plasma amplify magnetic fields ?
  - What are at the time and spatial scales on which this happens ?
  - At what amplitude do they saturate ? What field structure is produced ?
- A complex and multifaceted problem
  - Requires observations, phenomenology, theory, numerics and experiments

A touch of history

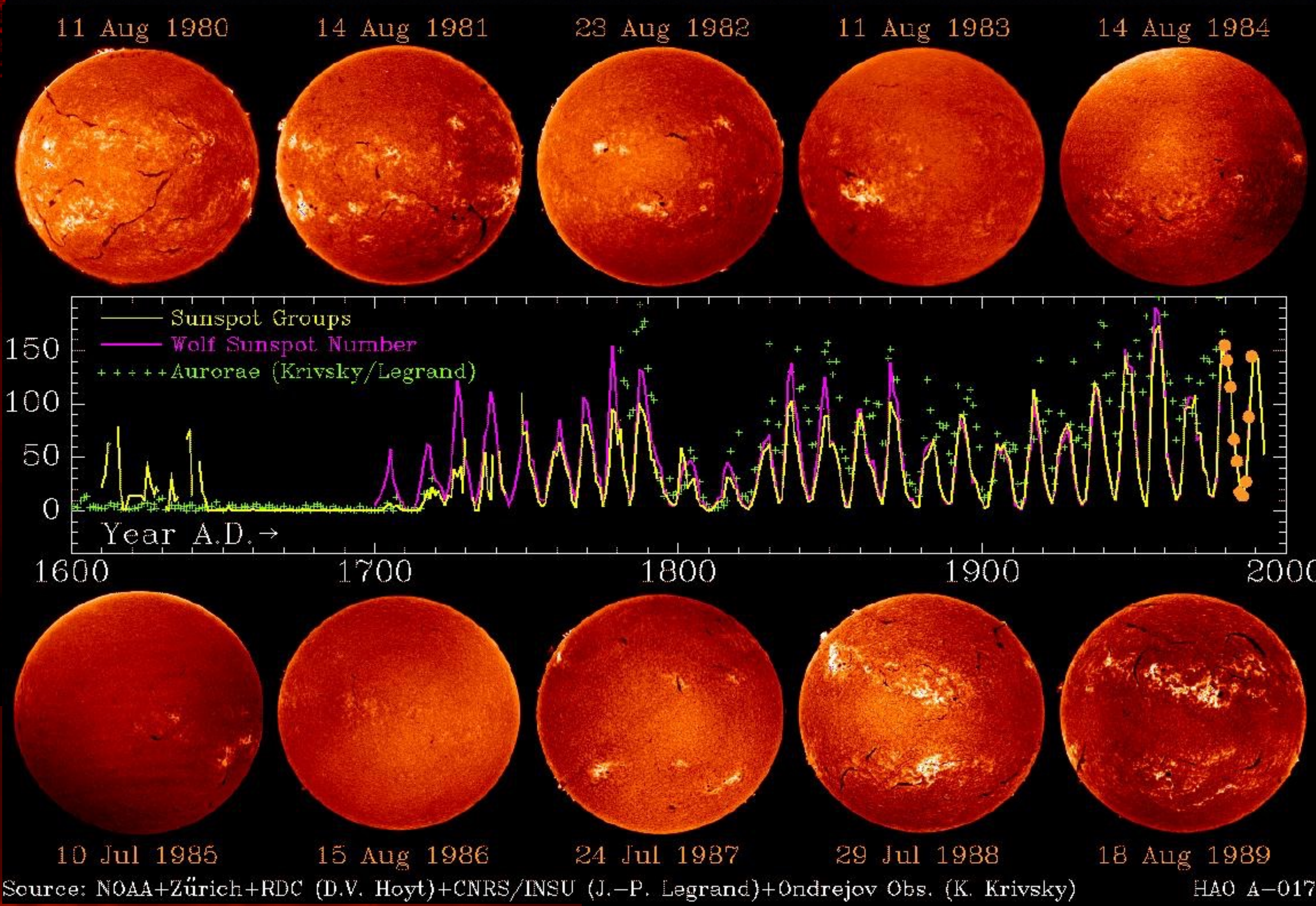
- Self-exciting fluid dynamos are now a century-old idea
  - First invoked by Larmor in 1919 (sunspot magnetism)

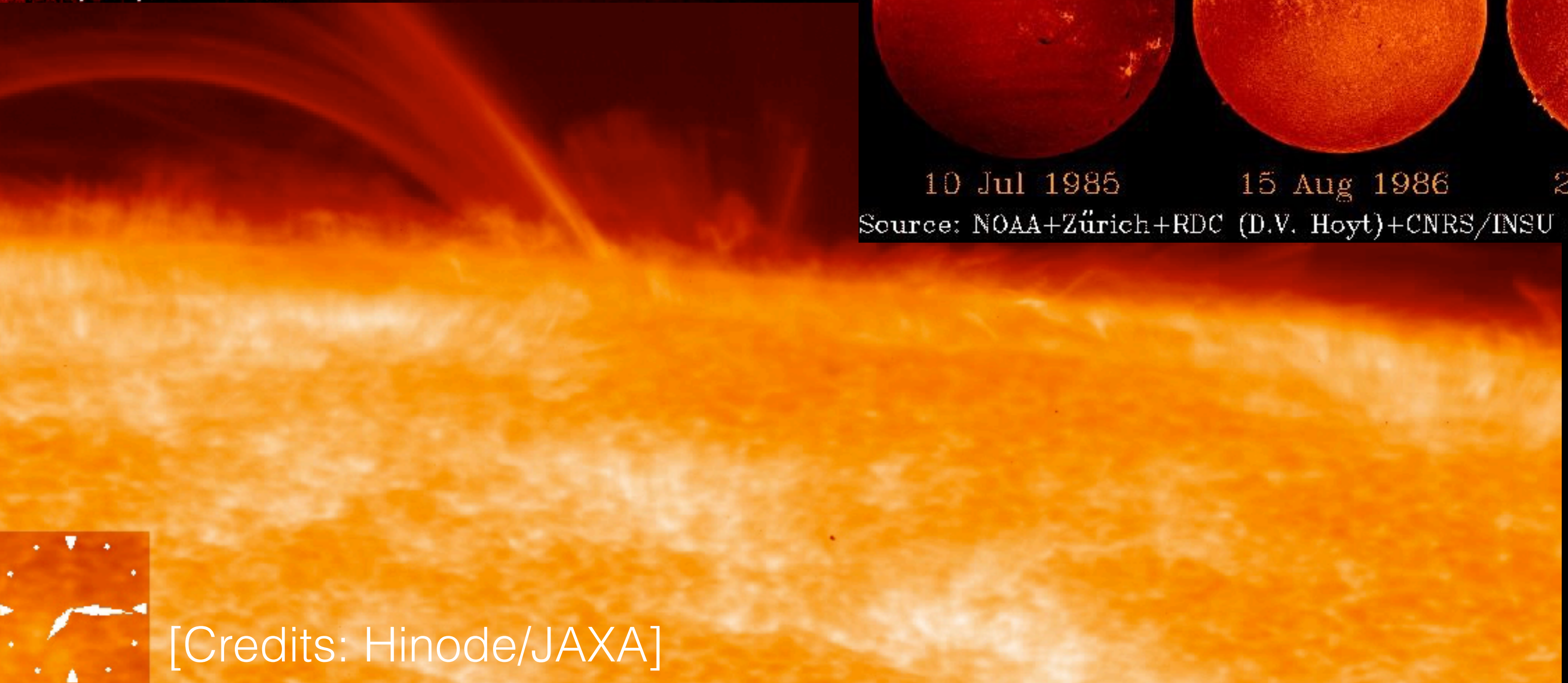
- The idea took a lot of time to gain ground
  - Cowling's antidynamo theorem (1933)
  - First examples in the 1950s (e.g. Herzenberg dynamo)
  - Parker's solar dynamo phenomenology (1955)
- Golden age of mathematical theory
  - Alpha effect / mean-field: Steenbeck, Krause, Raedler 1966, Moffatt, Roberts etc. (1970s)
  - Small-scale dynamo theory: Kazantsev 1967, Kraichnan, Zel'dovich et al. (70s-80s)
- Numerical and experimental era
  - Numerical evidence of turbulent dynamos: Meneguzzi et al. 1981, flourishing since then
  - Experimental evidence: Riga, Karlsruhe (~2000), VKS (2007), plasma underway (2005+)
  - Great observational radio and spectro-polarimetric prospects too (stellar, galactic, cosmo)

Solar magnetism



[Credits: SOHO/NASA]

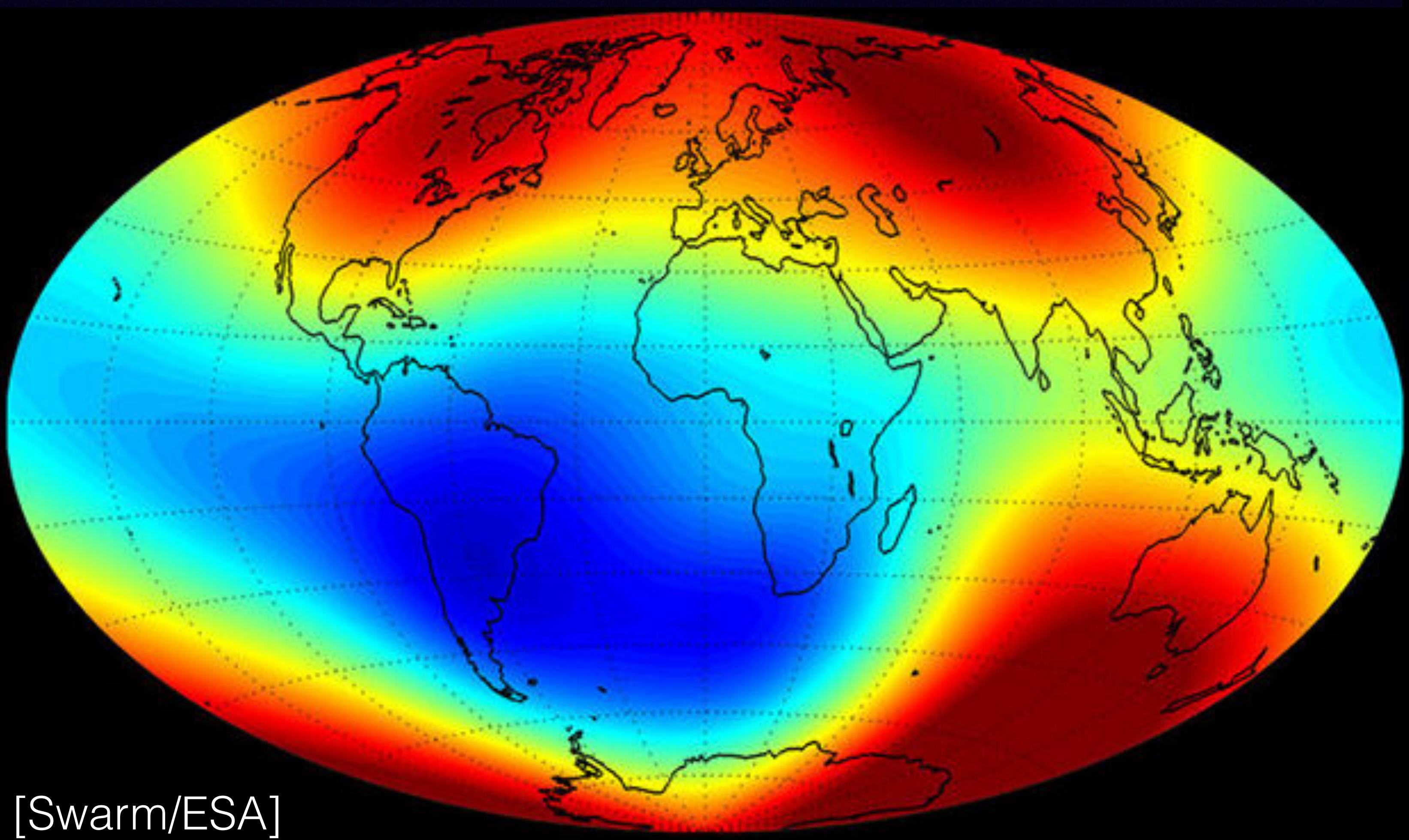



[Credits: Hinode/JAXA]

Global solar cycle dynamics
~ 1G-a few kG (sunspots)

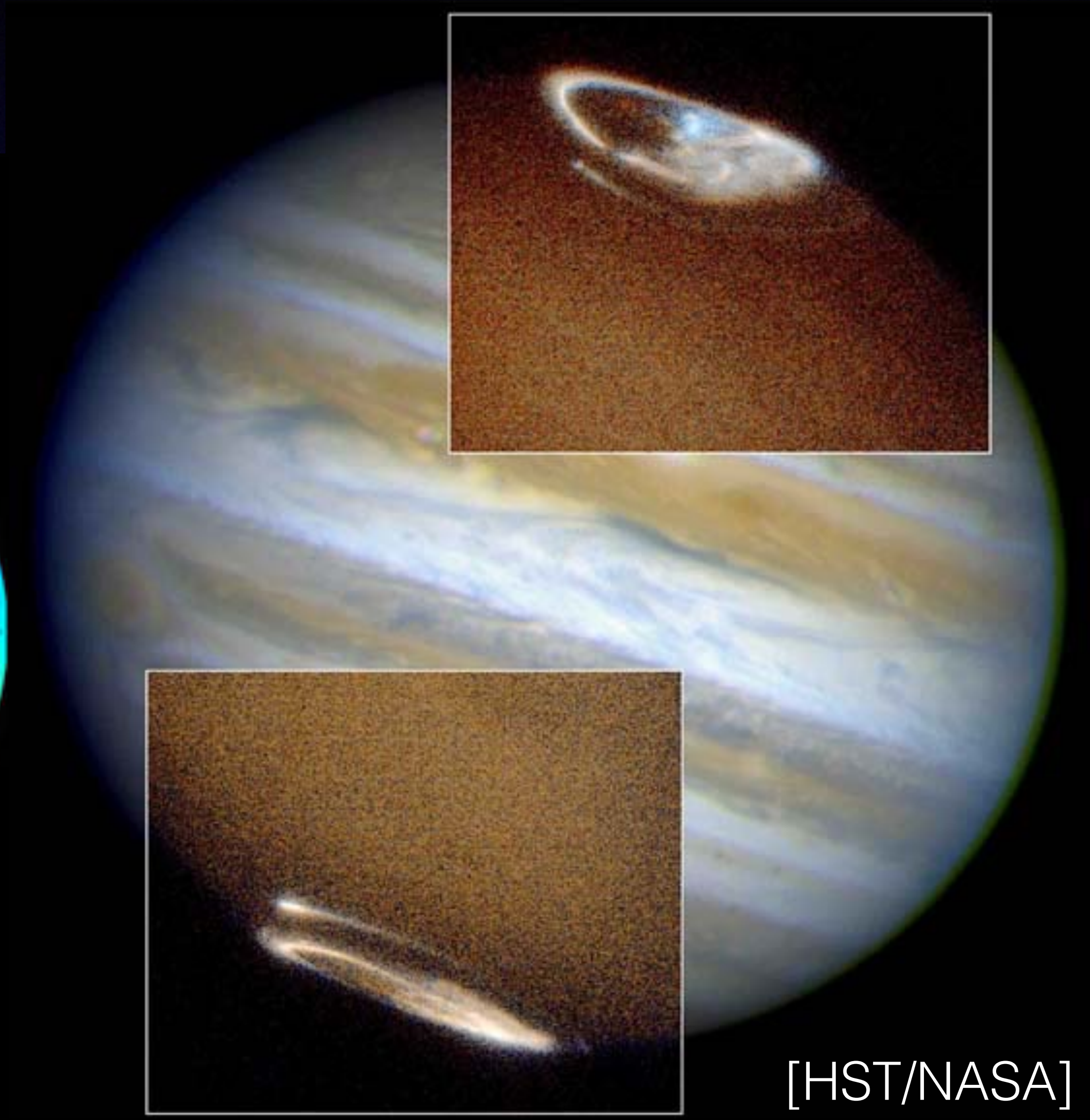
Small-scale surface dynamics
~ up to kG

Planetary magnetism



[Swarm/ESA]

Earth's magnetic field (2014) ~ 10-50 G
(0.1-0.5 G at the surface)

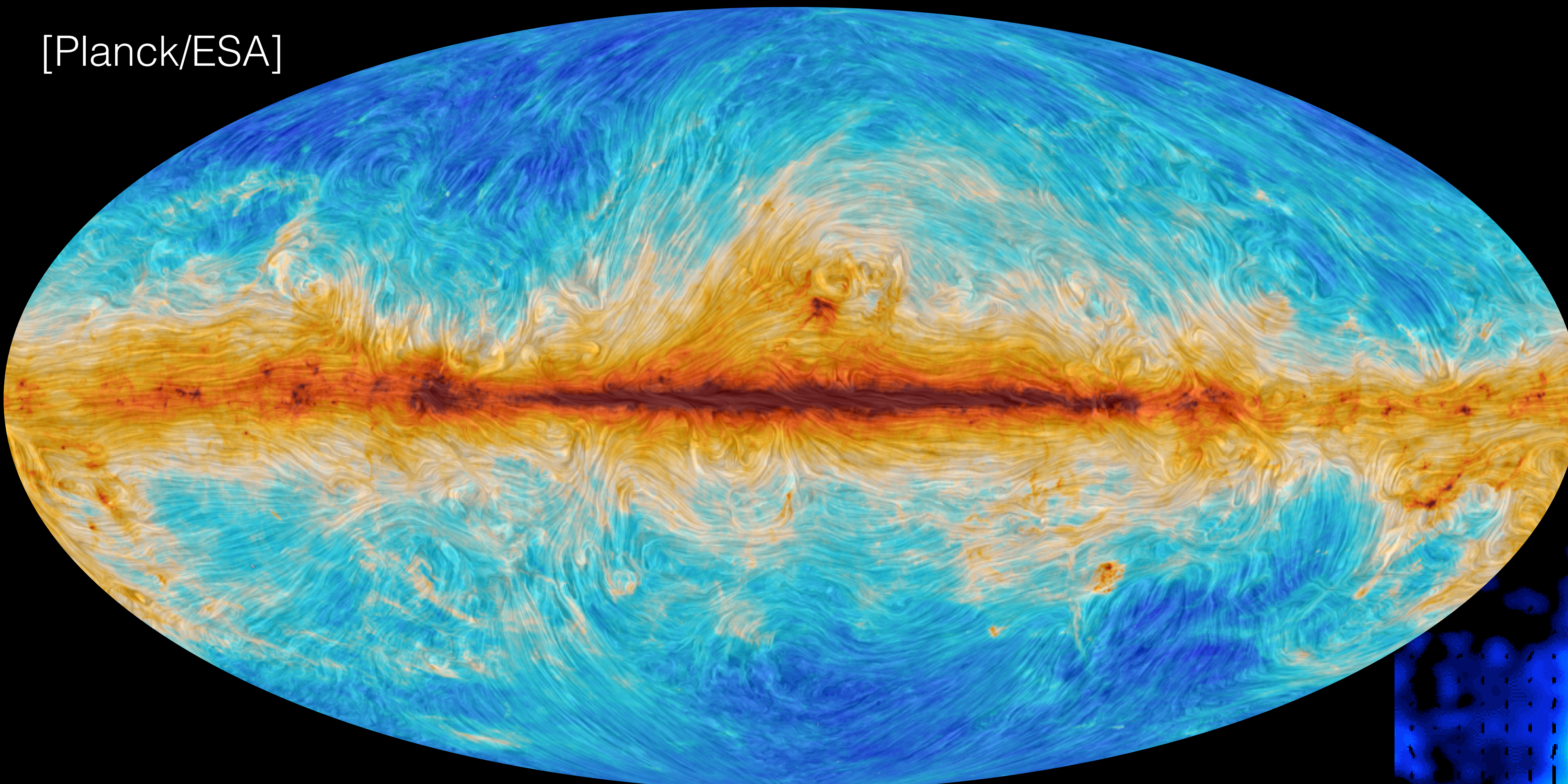


[HST/NASA]

Jupiter Auroras

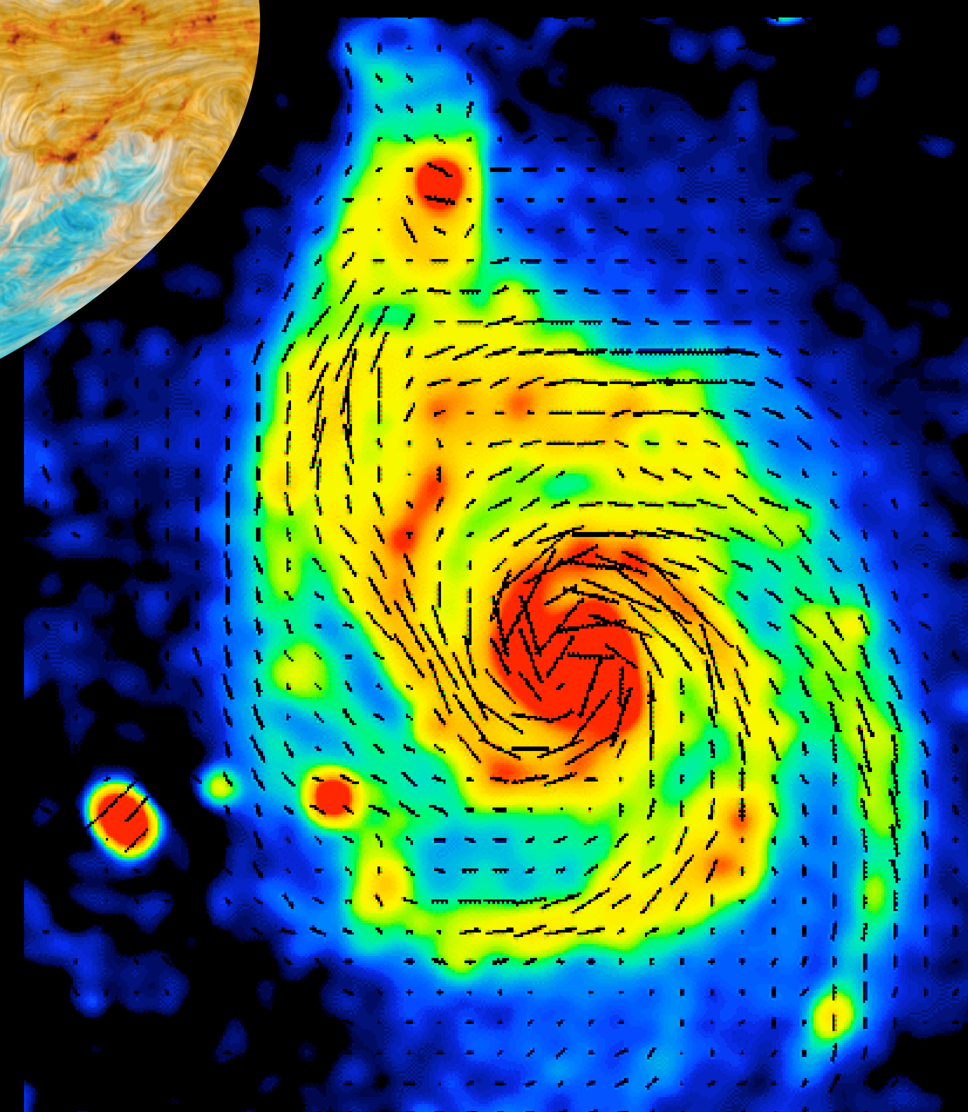
Galactic magnetism

[Planck/ESA]



Galactic magnetic field ~ 10 μG

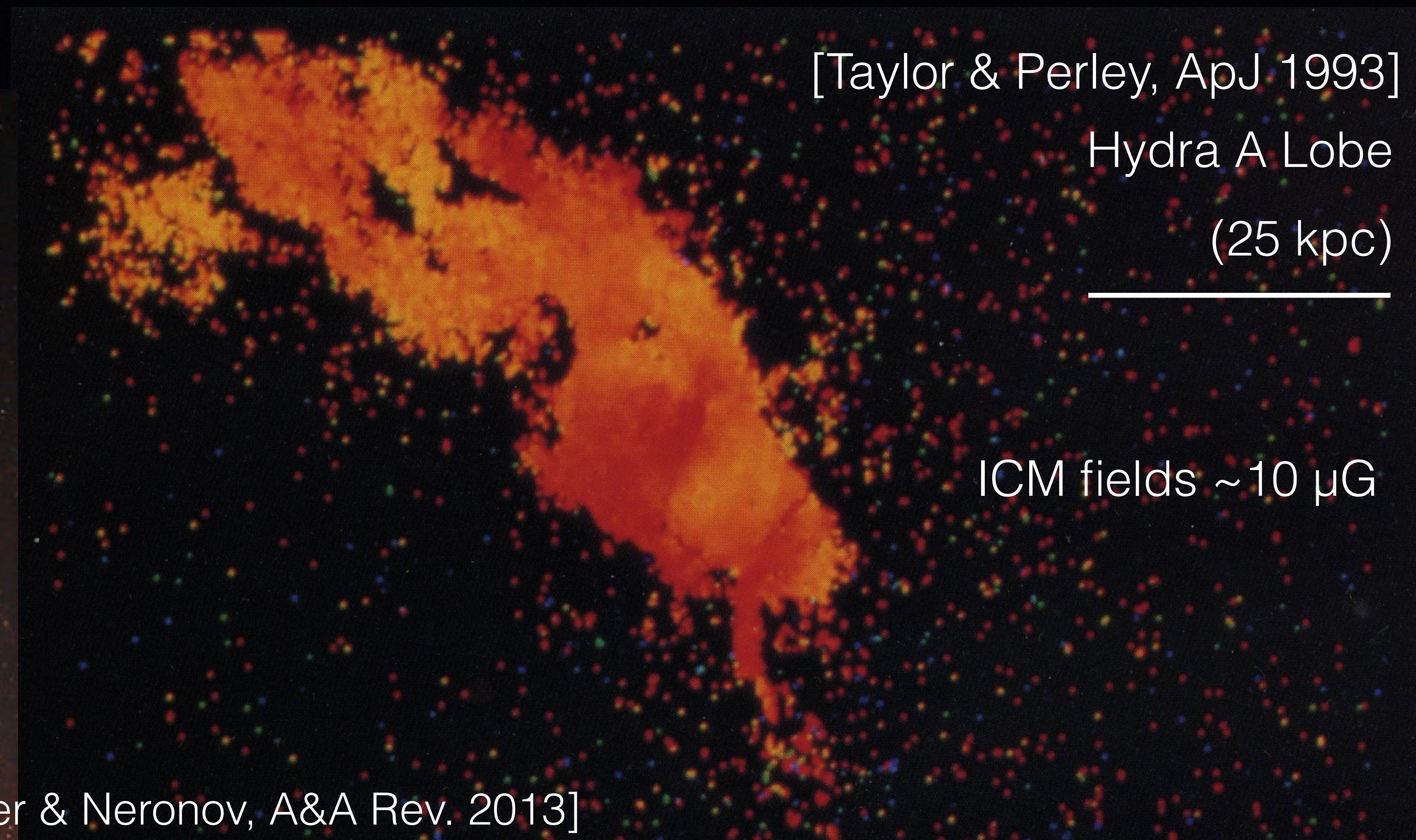
M51 magnetic field



[Beck et al. VLA/Effelsberg]

Galaxy clusters and cosmic magnetism

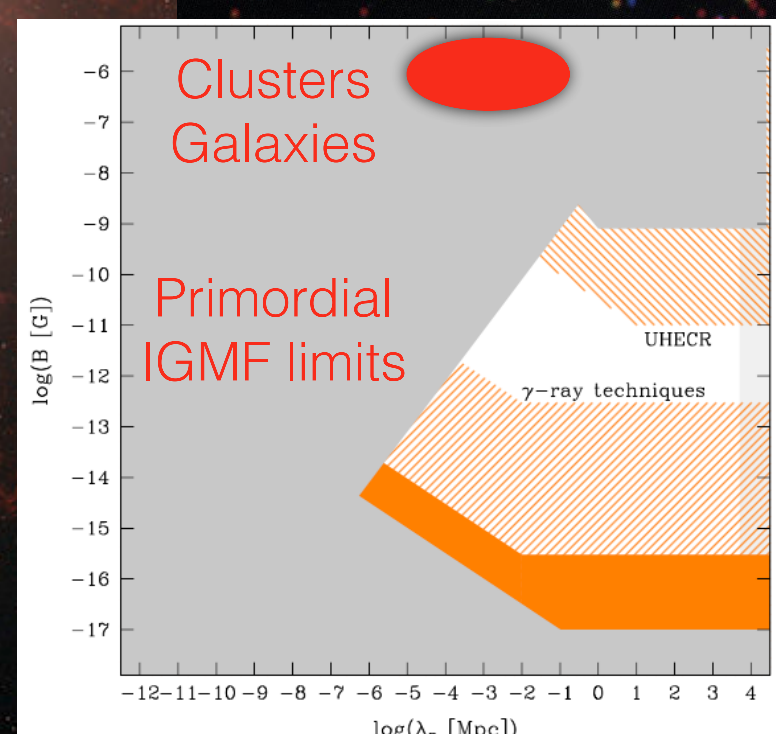
[Fabian et al. ESA/NASA]
Perseus/NGC 1275 filaments



[Taylor & Perley, ApJ 1993]
Hydra A Lobe
(25 kpc)

ICM fields ~10 μG

[Durrer & Neronov, A&A Rev. 2013]



Takeaway phenomenological points

- Many astrophysical objects have global, ordered fields
  - Differential rotation, global symmetries and geometry important
  - Coherent structures and MHD instabilities may also be very important
  - Motivation for the development of “large-scale” dynamo theories
- Lots of “small-scale”, random fields also discovered from the 70s
  - These come hand in hand with global magnetism
  - Simultaneous development of “small-scale dynamo” theory
- Astrophysical magnetism is in a nonlinear, saturated state
  - Linear theory likely not the whole story (or requires non-trivial justification)
  - Multiple scale interactions expected to be important

Setting the stage

Mathematical formulation

- Compressible, viscous, resistive MHD equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

Lorentz force

External forcing (spoon, gravity etc.)

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \frac{\mathbf{j} \times \mathbf{B}}{c} + \nabla \cdot \boldsymbol{\tau} + \mathbf{F}(\mathbf{x}, t)$$

Viscous stresses

$$\boldsymbol{\tau}_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{u} \right)$$

Electromotive force

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B})$$

Magnetic diffusion $\eta = \frac{c^2}{4\pi\sigma}$

$$\nabla \cdot \mathbf{B} = 0 \qquad \mathbf{j} = \frac{c}{4\pi} \nabla \times \mathbf{B}$$

Dissipation

Thermal diffusion

$$\rho T \left(\frac{\partial s}{\partial t} + \mathbf{u} \cdot \nabla s \right) = D_\mu + D_\eta + \nabla \cdot (\kappa \nabla T)$$

Magnetic field energetics

- Magnetic energy equation

$$\frac{d}{dt}\int \frac{\mathbf{B}^2}{8\pi} dV = -\int \mathbf{u}\cdot\frac{(\mathbf{j}\times\mathbf{B})}{c} dV - \frac{c}{4\pi}\oint (\mathbf{E}\times\mathbf{B})\cdot d\mathbf{S} - \int \frac{\mathbf{j}^2}{\sigma} dV$$

Minus the work of the Lorentz force on the flow

Poynting flux

Ohmic dissipation

  - Magnetic field is generated at the expense of kinetic energy

- Simple but enlightening local equation (ideal MHD)

$$\frac{1}{B}\frac{DB}{Dt} = \hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla\mathbf{u} - \nabla\cdot\mathbf{u}$$

Stretching rate

Compression rate

$$\hat{\mathbf{b}} = \frac{\mathbf{B}}{B}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla$$

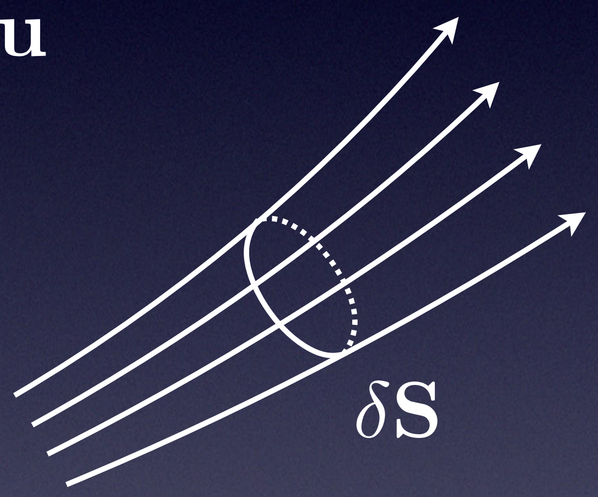
Conservation laws in ideal MHD

- Alfvén's theorem(s)
  - Magnetic field lines are "frozen into" the fluid just as material lines

$$\frac{D}{Dt}\left(\frac{\mathbf{B}}{\rho}\right) = \frac{\mathbf{B}}{\rho}\cdot\nabla\mathbf{u} \qquad\qquad \frac{D\delta\mathbf{r}}{Dt} = \delta\mathbf{r}\cdot\nabla\mathbf{u}$$

  - Magnetic flux through material surfaces is conserved

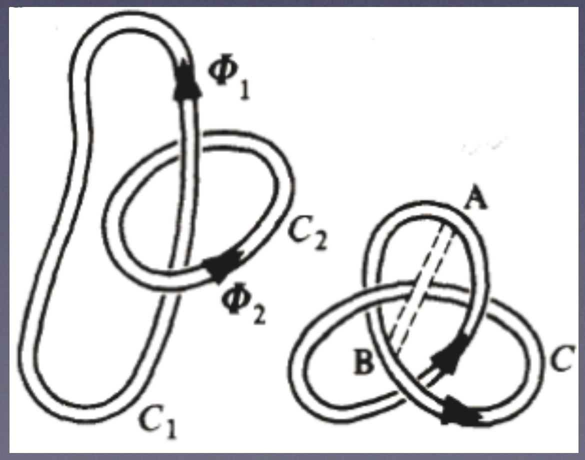
$$\frac{D}{Dt}(\mathbf{B}\cdot\delta\mathbf{S}) = 0$$



- Magnetic helicity $\mathcal{H}_m = \int \mathbf{A}\cdot\mathbf{B}\,d^3\mathbf{r}$ conservation
  - A measure of magnetic linkage / knottedness

$$\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} = -\mathbf{E} - \nabla\varphi$$

$$\frac{\partial}{\partial t}(\mathbf{A}\cdot\mathbf{B}) + \nabla\cdot[c\varphi\mathbf{B} + \mathbf{A}\times(\mathbf{u}\times\mathbf{B})] = 0$$



Simple MHD system for dynamo theory

- Incompressible, resistive, viscous MHD
  - Captures a great deal of the dynamo problem

Magnetic tension

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \mathbf{B} \cdot \nabla \mathbf{B} + \nu \Delta \mathbf{u} + \mathbf{f}(\mathbf{x}, t)$$

Induction/stretching

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} + \eta \Delta \mathbf{B}$$

$$P = p + \frac{B^2}{2}$$

Advection

Resistive diffusion

$$\nabla \cdot \mathbf{u} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

p and $\mathbf{B}$ rescaled by ρ and $(4\pi\rho)^{1/2}$

- Often paired with simple periodic boundary conditions
  - Can be problematic in some cases (more later)

Scales and dimensionless numbers

- System/integral scale ℓ_0, U_0
- Fluid system with two dissipation channels
  - Dimensionless numbers:

$$\mathrm{Re} = \frac{\ell_0 U_0}{\nu} \qquad \mathrm{Rm} = \frac{\ell_0 U_0}{\eta} \qquad \mathrm{Pm} = \frac{\nu}{\eta}$$

  - Kolmogorov viscous scale $\ell_\nu \sim \mathrm{Re}^{-3/4}\,\ell_0$, $u_\nu \sim \mathrm{Re}^{-1/4}\,U_0$
  - Magnetic resistive scale ℓ_η (Pm-dependent)
- Another important dimensionless quantity
  - Eddy turnover time $\tau_{\mathrm{NL}} \sim \ell_u/u$
  - Flow/eddy correlation time τ_c

$$\mathrm{St} = \frac{\tau_c}{\tau_{\mathrm{NL}}}$$

Strouhal/Kubo number

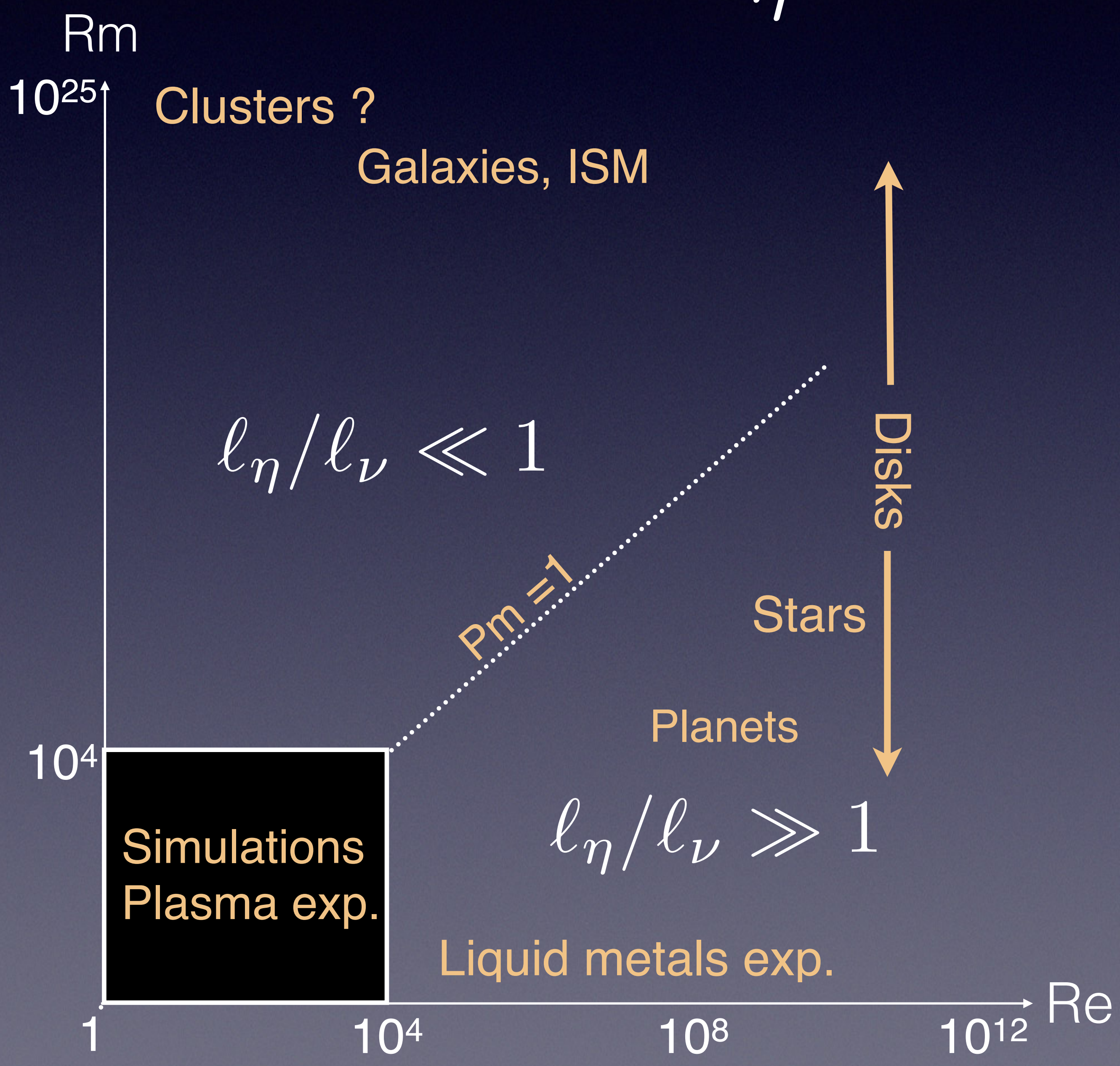
The magnetic Prandtl number landscape

- Wide range of Pm in nature
  - Liquid metals have Pm << 1
  - Computers have Pm ~ O(1)
- For a collisional hydrogen plasma [Te=Ti in K, *n* in S.I.]

$$\mathrm{Pm} = 2.5 \times 10^3 \frac{T^4}{n \ln \Lambda^2}$$

- Pm<1 and Pm>1 seemingly very different situations
  - Naively, Pm>1 makes life easier for magnetic fields

$$\mathrm{Pm} = \frac{\nu}{\eta}$$

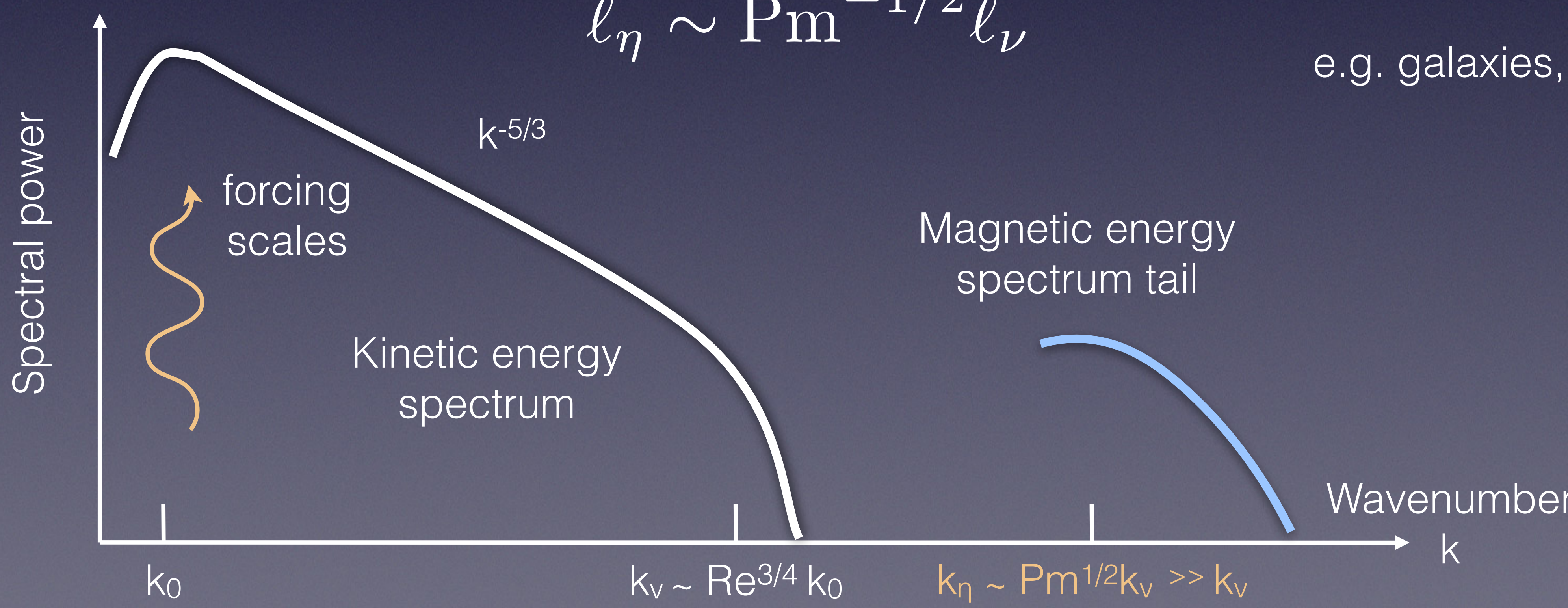


Large magnetic Prandtl numbers

- Pm > 1: resistive cut-off scale is smaller than viscous scale
  - In Kolmogorov turbulence, rate of strain goes as $\ell^{-2/3}$
  - Viscous eddies are the fastest at stretching B: $u_\nu / \ell_\nu \sim Re^{1/2}\, U_0 / \ell_0$
  - To estimate the resistive scale ℓ_η, balance stretching by these eddies $\sim u_\nu/\ell_\nu$ with ohmic diffusion rate η/ℓ_η^2

$$\ell_\eta \sim \mathrm{Pm}^{-1/2} \ell_\nu$$

e.g. galaxies, clusters

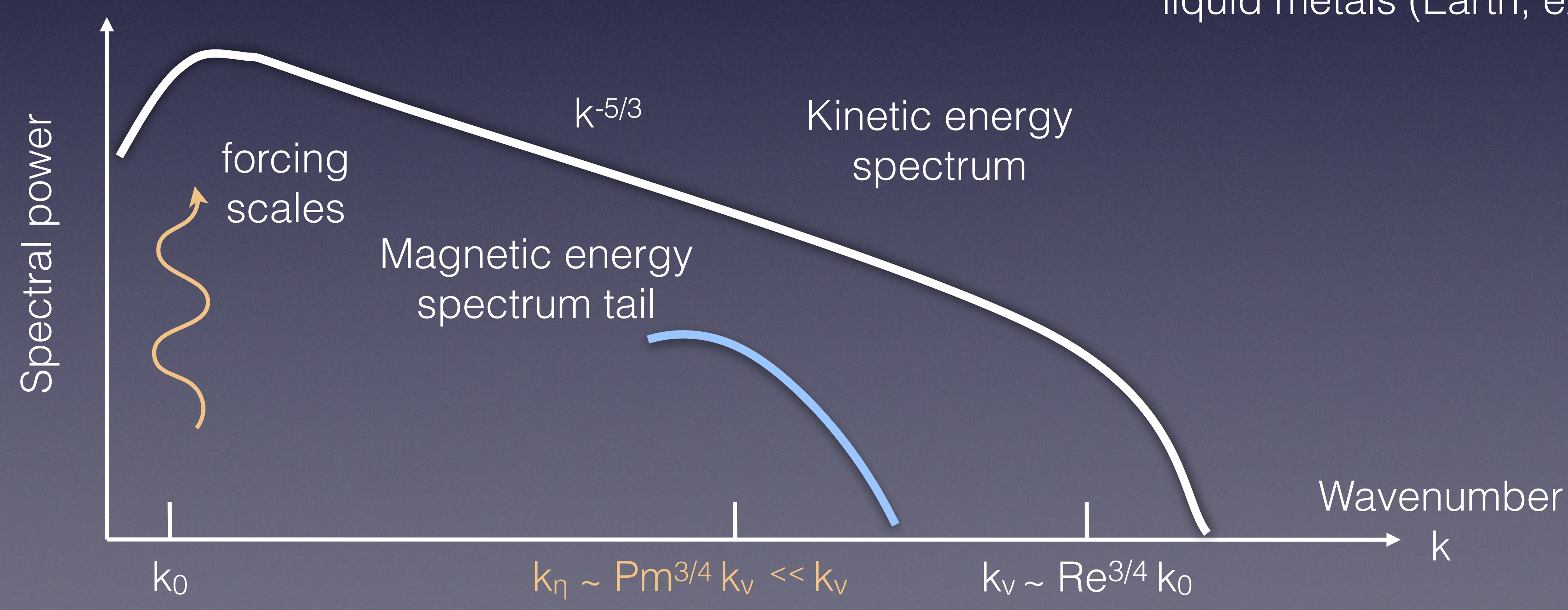


Low magnetic Prandtl numbers

- Pm < 1: resistive cut-off falls in the turbulent inertial range
  - To estimate the resistive scale ℓ_η, balance magnetic stretching by the eddies at the same scale ~ u_η/ℓ_η, with diffusion η/ℓ_η^2
  - i.e., $\mathrm{Rm}(\ell_\eta) = u(\ell_\eta)\,\ell_\eta / \eta \sim 1$

$$\ell_\eta \sim \mathrm{Pm}^{-3/4}\ell_\nu$$

e.g. stellar, solar,
liquid metals (Earth, experiments)

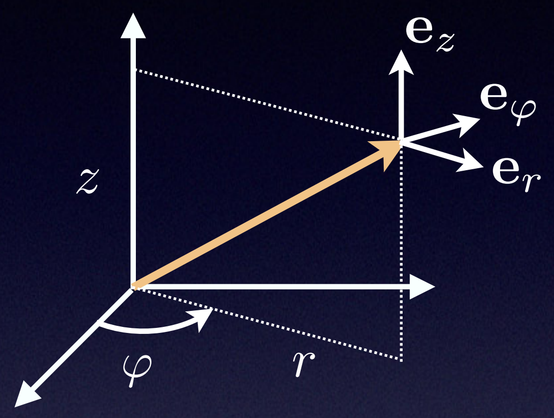


Dynamo fundamentals

- The problem of exciting a dynamo is an instability problem
  - Growth requires stretching to overcome diffusion (measured by $\mathrm{Rm} = \dfrac{\ell_0 U_0}{\eta}$)
- Kinematic dynamo problem: $\dfrac{\partial \mathbf{B}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{u} + \eta\Delta\mathbf{B}$
  - Find exponentially growing solutions of the linear induction equation (velocity field is prescribed)
- Dynamical problem considers effects of Lorentz force on $\mathbf{u}$
  - Saturated state of kinematic dynamos: non-linear magnetic back reaction
  - Subcritical scenarios: e.g. joint excitation of u and B via MHD instabilities
- Slow vs Fast
  - A dynamo is slow/fast if its growth rate does/doesn't vanish as $\eta \to 0$

Cowling's antidynamo theorem

- Axisymmetric dynamo action is impossible [Cowling, MNRAS, 1933]
  - In polar geometry, write



Poloidal　　Toroidal

- $\mathbf{B} = \nabla \times (\chi \mathbf{e}_\varphi / r) + r\psi\, \mathbf{e}_\varphi$
- $\mathbf{u} = \mathbf{u}_{\rm pol} + r\Omega\, \mathbf{e}_\varphi$

$$\frac{\partial \chi}{\partial t} + \mathbf{u}_{\rm pol} \cdot \nabla \chi = \eta \left(\Delta - \frac{2}{r}\frac{\partial}{\partial r} \right) \chi$$

No source term

$$\frac{\partial \psi}{\partial t} + \mathbf{u}_{\rm pol} \cdot \nabla \psi = \mathbf{B}_{\rm pol} \cdot \nabla \Omega + \eta \left(\Delta + \frac{2}{r}\frac{\partial}{\partial r} \right) \psi$$

- Poloidal flow can only redistribute flux so χ must decay ultimately
- As χ decays, so must the toroidal field

- Note: only applies if u and B share the same symmetry axis

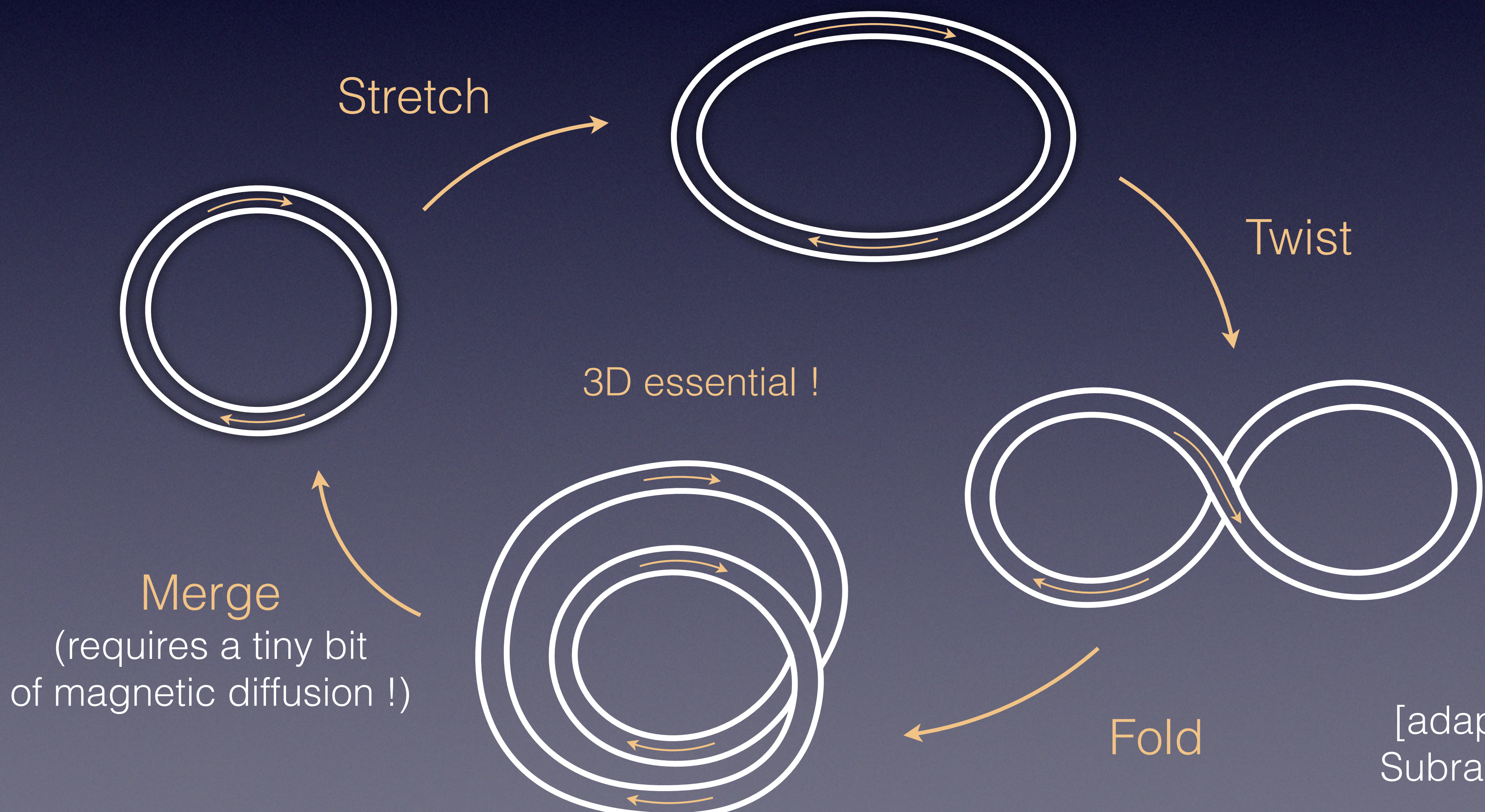
Antidynamo theorems and their implications

- Many other antidynamo results can be proven
  - Plane two-dimensional motions cannot sustain a dynamo [Zel'dovich's theorem, JETP 1957]
  - A purely toroidal flow cannot sustain a dynamo
  - $\mathbf{B}(x, y, t)$ cannot be a dynamo field
- Dynamos are only possible in "complex" geometries or flows
  - An extra burden for both theory and numerics
  - A popular "minimal" configuration is 2.5D (or 2D-3C)
    - $\mathbf{u}(x, y, t)$ with all three components non-vanishing
    - $\mathbf{B}(x, y, z, t) = \mathcal{R}\left\{\mathbf{b}(x, y, t)e^{ik_z z}\right\}$

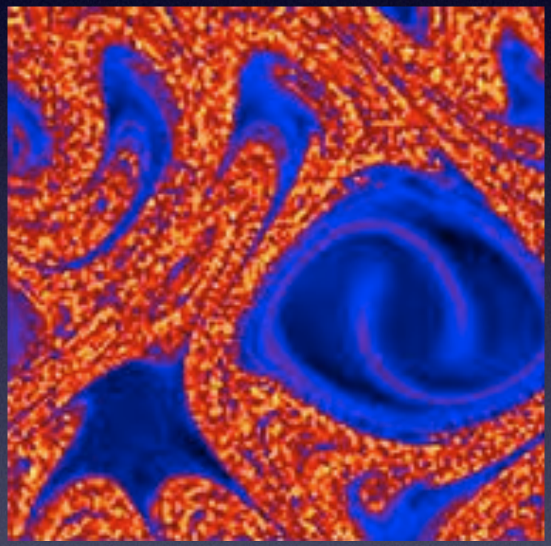
The fast dynamo paradigm

[Vainshtein & Zel'dovich, SPU, 1972]

- Chaotic stretching, twisting, folding and merging of field lines
  - For small diffusion, field doubles at each "iteration" (characteristic time)
  - Exponential growth with "ideal" growth rate $\gamma_\infty = \ln 2 \sim$ stretching rate



[adapted from Brandenburg & Subramanian, Phys. Rep. 2005]



Lyapunov exponents of Galloway-Proctor flow [credits F. Cattaneo]

An imperfect dichotomy

- Large-scale dynamo effect
  - Magnetic field generated on
    - long system time (Ω^{-1}, S^{-1}), spatial scales (L) much larger than flow scales ℓ_0
    - also lots of magnetic fluctuations on low and sub flow scales down to the magnetic resistive scale
- Small-scale dynamo effect
  - Magnetic field generated on short time (ℓ/u), spatial scales (ℓ) from flow scales down to the resistive scale
- Each of these can be excited by laminar or turbulent flows
- They have traditionally mostly been described by different theories
  - in all MHD astrophysical settings, large-scale dynamos are swamped by small-scale ones
  - this creates a lot of theoretical difficulties
- MHD instabilities also play a key role in large-scale dynamos
  - the magneto-rotational, and other magnetoshear instabilities
  - Kelvin-Helmholtz instability coupled to magnetic buoyancy