Blind Quantum Process Tomography

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Abstract

Quantum process tomography (QPT) methods aim at identifying a given quantum process. QPT is a major quantum information processing tool, since it especially allows one to characterize the actual behavior of quantum gates, which are the building blocks of quantum computers. We aim at developing blind methods of QPT i.e. methods that do not require the user to know beforehand what qubit are used to identify the process.

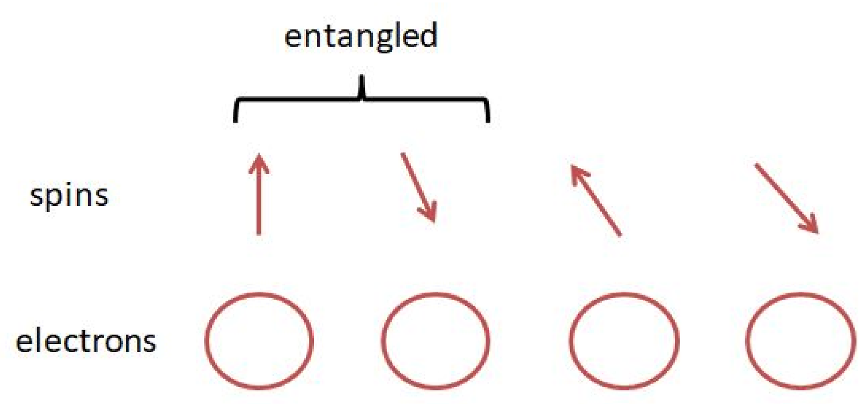
Introduction to Quantum Systems

Classical register

1	0	1	0

- State: $B = (b_1 \dots b_n) \in \{0,1\}^n$
- Reading: Straightforward
- Evolution: Does not evolve

Quantum register



- State:
  - The state is a superposition of 2^n kets: $\Phi_1(t)\,|0\dots0\rangle + \Phi_2(t)\,|0\dots01\rangle + \dots + \Phi_{2^n}(t)\,|1\dots1\rangle$
  - Characterized by $\Phi(t) = \begin{pmatrix}\Phi_1(t)\\ \vdots \\ \Phi_{2^n}(t)\end{pmatrix} \in \mathcal{C}^{2^n}, \Phi(t)'\Phi(t) = 1$
  - The modulus of each component contains information on the likelihood of each outcome when a measurement is made
  - The relative phases are important when measuring the state in a non-canonical base or when considering the evolution of the system
  - The gobal phase has no physical meaning, thus we generally assume that the first component is a positive real number
- Reading:
  - Measuring $\phi(t)$ gives a random set of n bits $\mathcal{M}_\phi \in \{0,1\}^n$
  - If the measurement is made in the canonical base: $p(\mathcal{M}_\phi = B_k) = |\Phi_k(t)|^2$ where B_k the binary expression of k and $|.|^2$ is the component-wise square modulus.
  - If $\phi(t)$ is measured in an orthonormal base represented by Q, then the 2^n dimensional vector containing the probabilities of each bit string in the order is: $p(\mathcal{M}_{\mathcal{Q}_\phi}) = |Q\Phi(t)|^2$
- Evolution:
  - Ruled by the Schrödinger equation: $\frac{\partial\phi(t)}{\partial t} = \frac{\mathcal{H}}{i\hbar}\phi(t)$. Where $\mathcal{H}$ is the Hamiltonian and $\hbar$ the reduced plank constant.
  - In a closed system, if the Hamiltonian is time invariant $\phi(t+\Delta_t) = M\phi(t)$ with $M = e^{\Delta_t \frac{\mathcal{H}}{i\hbar}}$. $\mathcal{H}$ is hermitian so M is unitary ($M'M = I_{2^n}$)

Main Objectives

- We assume a constant Hamiltonian. We want to estimate the unitary matrix M that characterizes the process by measuring the output corresponding to unknown input. M can only be determined up to an unknown global phase.
- Standard methods of quantum process tomography aims at identifying the process by measuring its effect on a known set of input state.

$$\Phi_1 \dots \Phi_k \xrightarrow{\Delta_t} M\Phi_1 \dots M\Phi_k$$

known measured

- We aim to estimate M by measuring the evolution of one or several unknown state going through the system several times times.

$$\Phi_0 \xrightarrow{\Delta_t} M\Phi_0 \xrightarrow{\Delta_t} M^2\Phi_0 \xrightarrow{\Delta_t} \dots \xrightarrow{\Delta_t} M^k\Phi_0$$

- At each step, we assume that enough measurement are made to achieve quantum state tomography (QST) i.e. find the quantum state up to a global phase.
- Two complementary approaches are considered to find M
  1. A least square approach that uses the states estimated by the QST to find a matching unitary M.
  2. A maximum likelihood approach that model the measurement error to find the M and initial state Φ_0 that are the most likely.

Measurement and quantum state tomography

Quantum state tomography aims at estimating a quantum state averaging multiple measurement in several bases. We are trying to achieve this using only measurements in 3 bases noted X, Y and Z;

- The bases are called X, Y, and Z because if the qubits represent the spins of electrons, then the bases represent a spin measurement along the X, Y and Z direction.
- They are represented by the orthonormal matrices Q_{X_n} Q_{Y_n} and Q_{Z_n} (n is the number of qubit)
- Q_{Z_n} is the identity matrix I_{2^n}, and one can show that Q_{X_n} and Q_{Y_n} are defined thusly:
  $Q_{X_1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, Q_{X_{n+1}} = \frac{1}{\sqrt{2}}\begin{pmatrix}Q_{X_n} & Q_{X_n}\\ Q_{X_n} & -Q_{X_n}\end{pmatrix}, Q_{Y_1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}, Q_{Y_{n+1}} = \frac{1}{\sqrt{2}}\begin{pmatrix}Q_{Y_n} & -iQ_{Y_n}\\ Q_{Y_n} & iQ_{Y_n}\end{pmatrix}$
- By making N measurement on each basis and averaging the frequency of occurrence of every bit string we get $\overline{\mathcal{M}_{\mathcal{X}\Phi}}$, $\overline{\mathcal{M}_{\mathcal{Y}\Phi}}$ and $\overline{\mathcal{M}_{\mathcal{Z}\Phi}}$
- Their expected values are respectively $|Q_{X_n}\Phi|^2$, $|Q_{Y_n}\Phi|^2$ and $|\Phi|^2$
- Our aim is to find Φ up to a global phase using these measurement. Obviously the information in $\overline{\mathcal{M}_{\mathcal{Z}\Phi}}$ gives us the modulus of Φ.
- We have yet to prove that the 3 measurements are enough to recover the relative phases of Φ for $n \geqslant 3$.

Least Square

- Our first method to estimate M uses the estimates $\widehat{\Phi_j} \forall j \in \{1, ..., k\}$ of $\Phi_j = M^j\phi_0$.
- From there we find a unitary matrix M that minimizes the square norm of $\epsilon_{LS} = M(\Phi_1, ..., \Phi_{k-1}) - (\Phi_2, ..., \Phi_k)$
- Despite the name of the section the least square algorithm is not ideal because it would not guarantee a unitary solution
- Fortunately this is a well known problem in the aerospace community and we have an analytical solution for M
- The only problem is that all the Φ_j can only be estimated up to a global phase, and though the global phase of $(\Phi_1, ..., \Phi_k)$ has no unwanted impact on the estimate of M, the phase between the columns do. We had to develop methods to identify those phases.

Maximum likelihood

Once the least square method gives us a first analytical estimate of M we have a good initial point for a maximum likelihood estimation.

- We first have to define the random error on the measurement $\mathcal{M}$ that contains the measurement on X, Y and Z for all k state
  - Each component of $\mathcal{M}$ is the frequency of apparition of a sequence of bit for a given measurement. Thus the elements of $N\mathcal{M}$ are random variables following binomial laws.
  - Asymptotically ($N \to +\infty$), $\mathcal{M}$ is a gaussian vector. Its covariance matrix ($\Sigma_\mathcal{M}$) can be estimated using only $\mathcal{M}$
  - The expected value of $\mathcal{M}$ depends only on Φ_0 and M it is noted $\mathcal{M}_{th}(\Phi_0, M)$, it is the reshaped version of $\left|\begin{pmatrix}Q_{X_n}\\ Q_{Y_n}\\ Q_{Z_n}\end{pmatrix} \times \begin{pmatrix}M\phi_0 & \dots & M^k\phi_0\end{pmatrix}\right|^2$ on a single column.
- The normalized log-likelihood of $\mathcal{M}$ is $\mathcal{L}(\Phi_0, M) = \big(\mathcal{M} - \mathcal{M}_{th}(\Phi_0, M)\big)'\Sigma_\mathcal{M}^{-1}\big(\mathcal{M} - \mathcal{M}_{th}(\Phi_0, M)\big)$
- We find the $\widehat{M}$, $\widehat{Phi_0}$ that maximizes the likelihood. Only $\widehat{M}$ is of interest to us
- If we consider that the initial state is not entangled, the likelihood maximization of the likelihood has $2n$ parameters for Phi_0 and 4^n parameters for M, so $4^n + 2n$ parameters in total, it scales really badly with the number n of qubit.

Future work

- The quantum state tomography is a core part of our algorithm, there is room for improvement there:
  - QST is a very well documented problem but the context considered in the literature is generally different (mixed states, different measurements).
  - For a number of qubit greater than 2 we do not have an analytical solution and must rely on multiple optimizations initialized at random to find a solution.
- For a high number n of qubit, the matrix M we try to estimate is huge ($2^n \times 2^n$) but in physical system it is generally sparse because a qubit only interacts with a limited number of neighbors. We could adapt our method to take the sparsity of M into account in high dimensions.
- In general, the QPT methods that can be found in the literature are not blind and require more types of measurements but they apply to open system and mixed states.
  - Mixed states are statistical mixture of pure states they have $4^n - 1$ parameters instead of $2^n - 2$.
  - Open system interact with they environment. Pure state are not suited to represent the state of an open system.
  - In an open system, even with a time invariant Hamiltonian, a process is not always unitary.
- We will try to adapt our method to mixed states and open system but probably keep the unitary model for M

References

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