Sparse Inverse Problems with Positivity Constraints in Earth and Space Sciences

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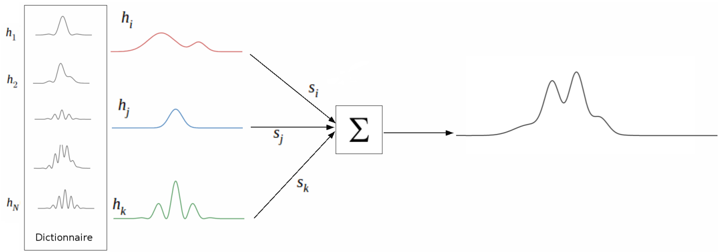
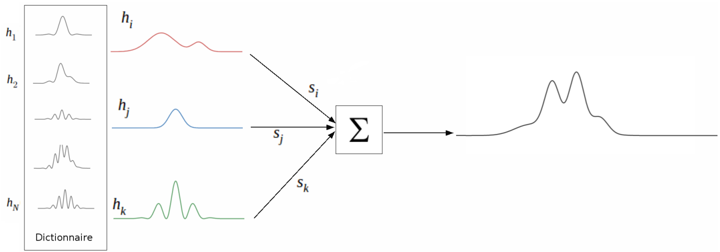
Sparse Inverse Problems



$$\underline{\text{Model :}}\ \boldsymbol{y} = \mathbf{H}\boldsymbol{x} + \epsilon \implies \underline{\text{Cost Function :}}\ J(\boldsymbol{x}) = \|\boldsymbol{y} - \mathbf{H}\boldsymbol{x}\|^2 + \alpha \|\boldsymbol{x}\|_0$$

Sparse Inverse Problems



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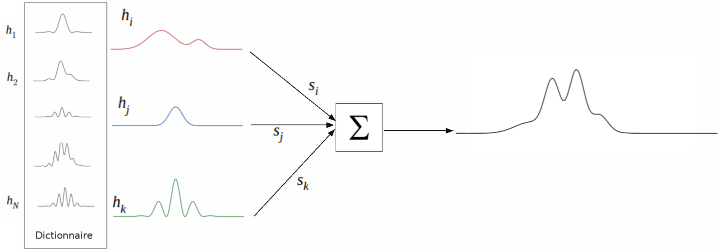

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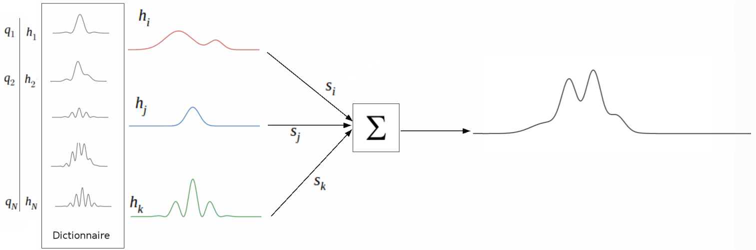
- Convex relaxation ($\ell_0 \rightarrow \ell_1$)
- Greedy algorithms

} ✗ sub optimal

- Bernoulli-Gaussian Model ✓optimal?



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Bernoulli-Gaussian Model (*Cheng et al* [3])

$$\begin{cases} q_k \in \{0, 1\} \\ \Pr(q_k = 1) = \lambda \end{cases} \quad \text{and} \quad \begin{cases} x_k | q_k = 1 & \sim & \mathcal{N}(0, \sigma_x^2) \\ x_k | q_k = 0 & \sim & \delta(x_k) \end{cases}$$

Posterior Mean Estimator

- Posterior Distribution (Bayes Rule) :

$$\overbrace{P(\boldsymbol{x},\boldsymbol{q},\lambda|\boldsymbol{y})}^{\text{Posterior}} \propto \overbrace{\exp\left(-\frac{1}{2\sigma_\epsilon^2}||\boldsymbol{y}-\mathbf{H}\boldsymbol{x}||^2\right)}^{\text{Likelihood}} \overbrace{P(\boldsymbol{x}|\boldsymbol{q})P(\boldsymbol{q}|\lambda)P(\lambda)}^{\text{Priors}}$$

$$\propto \mathcal{N}(\boldsymbol{x};\boldsymbol{\mu},\boldsymbol{\Gamma})f(\boldsymbol{q},\lambda)$$

- MCMC Methods - Gibbs Sampler

Gibbs Sampler

for each iteration n
    for k from 1 to K
        Sample $q_k|x_k$: Bernoulli distribution
        Sample $x_k|q_k$: Gaussian distribution
    end
end

Collapsed Gibbs Sampler

Marginalization (*M.Boudineau* Phd Thesis [2])

$$P(\boldsymbol{q}, \lambda | \boldsymbol{y}) \propto f(\boldsymbol{q}, \lambda) \int_{-\infty}^{\infty} \mathcal{N}(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{\Gamma}) dx \propto f(\boldsymbol{q}, \lambda)$$

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with Marginalization

for each iteration n
    for k from 1 to K
        Sample q_k : Bernoulli distribution
    end
end
Sample $x|q$: Gaussian distribution

✓ Fast convergence.

Boundary Constraints

Example : positivity prior, $x_k \geq 0 \ \forall k$

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Positive Bernoulli-Gaussian Model :

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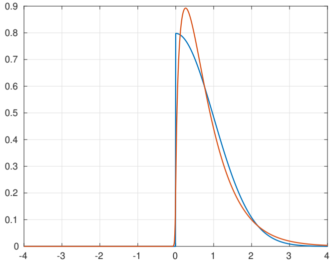
Issue :

✗ Marginalization is not possible !

✗ Very slow algorithms

Proposed solution :

Approximation by the Generalized Hyperbolic distribution [1] :



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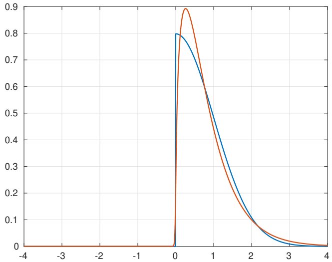

Scale and Location Mixture of Gaussians :

$$GH(x;\lambda,\alpha,\beta,\delta,\mu) = \int_0^\infty \mathcal{N}(x;\mu+\beta w, w)\, GIG(w;\lambda,\gamma,\delta)\, dw$$

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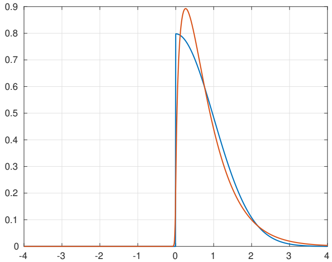

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✓ Possible to marginalize ★ Fast convergence.

References

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Any questions ?!